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# Gauge invariance and variational trivial problems on the bundle of connections

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**Abstract**

Given a principal bundle $P \to M$ we classify all first order Lagrangian densities on the bundle of connections associated to $P$ that are invariant under the Lie algebra of infinitesimal automorphisms. These are shown to be variationally trivial and to give constant actions that equal the characteristic numbers of $P$ if $\dim M$ is even and zero if $\dim M$ is odd. In addition, we show that variationally trivial Lagrangians are characterized by the de Rham cohomology of the base manifold $M$ and the characteristic classes of $P$ of arbitrary degree.


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## 1. Introduction

Let $p : C \to M$ be the bundle of connections of a principal $G$-bundle $\pi : P \to M$. The goal of this paper is twofold. First, we classify the Lagrangians on $J^1C$ which are invariant under the natural lift to $\mathfrak{X}(J^1C)$ of the full Lie algebra of infinitesimal automorphisms of $P$ (see Theorem 1). A classical result due to Utiyama (see [3,7,9,14,20]) states that a Lagrangian $\mathcal{L} : J^1C \to \mathbb{R}$ is gauge invariant if and only

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if it factors through the curvature mapping $\Omega : J^1C \to \bigwedge^2 T^*M \otimes \operatorname{ad} P$, i.e., $\mathcal{L} = \overline{\mathcal{L}} \circ \Omega$, by means of a differentiable function $\overline{\mathcal{L}} : \bigwedge^2 T^*M \otimes \operatorname{ad} P \to \mathbb{R}$ which, in turn, must be invariant under the adjoint representation of the Lie algebra $\mathfrak{g}$ of $G$ on the fibers of the adjoint bundle $\operatorname{ad} P$.

Due to the interest of the geometric version of Utiyama's theorem in gauge field theory, it seems reasonable to ask for the invariance under the group of all (not necessarily $\pi$-vertical) automorphisms. Within the framework of Riemannian vector bundles, i.e., when $G = U(k)$ or $G = O(k)$, partial results in this direction can been found in [10, Theorem 2.6.2] for polynomial Lagrangian densities but we should emphasize that the characterization given here in Theorem 1 is obtained in full generality as we work on the bundle of connections of an arbitrary principal bundle and we do not assume $L$ to be a polynomial. Unlike the Utiyama classification, invariance under the Lie algebra $\operatorname{aut} P$ of all infinitesimal automorphisms leads to variationally trivial Lagrangians. In fact, they are associated to Weil polynomials whose degree is half of the dimension of $M$, if $\dim M$ is even, and zero if $\dim M$ is odd (see Section 3.2). The action functional determined by these Lagrangians is constant and produces some characteristic numbers of the bundle. Thus, the group of all automorphisms provides a variational characterization of certain topological invariants of the bundle, which are of great importance in the topological theory of quantized fields.

Secondly, motivated by the aforementioned result, in Section 4.2 we characterize the gauge-invariant Lagrangian densities on $J^1C$ which are variationally trivial (or null Lagrangians); see Theorem 2. The structure of such densities is closely related to 'divergence-free' $S^d(\mathfrak{g}^*)^G$-valued multi-vector fields on $M$, where $S^d(\mathfrak{g}^*)^G$ stands for the coadjoint invariant polynomials of degree $d = 0, \ldots, [n/2]$, $n = \dim M$. It is shown that these Lagrangians are characterized by the de Rham cohomology of the base manifold $M$ and the characteristic classes of $P$ of arbitrary degree.

Certain results of the present paper can be considered as a strong generalization to an arbitrary Lie group of the results in [5] for classical electromagnetism.

## 2. Preliminaries

### 2.1. Automorphisms of a principal bundle

Consider a principal $G$-bundle $\pi : P \to M$. An *automorphism* $\Phi : P \to P$ is a diffeomorphism such that $\Phi(u \cdot g) = \Phi(u) \cdot g$, $\forall u \in P$, $\forall g \in G$. The set $\operatorname{Aut} P$ of all automorphisms of $P$ is a group under composition. An automorphism $\Phi \in \operatorname{Aut} P$ induces a unique diffeomorphism $\phi : M \to M$ on the base manifold $M$, characterized by $\pi \circ \Phi = \phi \circ \pi$. If $\phi$ is the identity map, then $\Phi$ is said to be a *gauge transformation*. The set of gauge transformations is a normal subgroup $\operatorname{Gau} P \subset \operatorname{Aut} P$, which is called the gauge group of $P$. In the case of the trivial bundle $pr_1 : M \times G \to M$, every automorphism $\Phi$ can be written as $\Phi(x, g) = (\phi(x), \psi(x) \cdot g)$, $x \in M$, $g \in G$, where $\phi : M \to M$ is a diffeomorphism and $\psi : M \to G$ is a differentiable map.

### 2.2. Invariant vector fields

Let $\mathfrak{X}(P)$ be the Lie algebra of vector fields on $P$. A vector field $X \in \mathfrak{X}(P)$ is said to be $G$-invariant if $TR_g \circ X \circ R_{g^{-1}} := (R_g)_*X = X$, $\forall g \in G$, where $R_g$ denotes the right action of $G$ on $P$. If $\Phi_t$ is the flow of a vector field $X \in \mathfrak{X}(P)$, then $X$ is $G$-invariant if and only if $\Phi_t \in \operatorname{Aut} P$, $\forall t \in \mathbb{R}$. Accordingly, we think of $G$-invariant vector fields as being the 'Lie algebra' of $\operatorname{Aut} P$ and we denote it by $\operatorname{aut} P \subset \mathfrak{X}(P)$. Each $G$-invariant vector field on $P$ is $\pi$-projectable. Similarly, a $\pi$-vertical vector field $X \in \mathfrak{X}(P)$ is $G$-invariant if and only if $\Phi_t \in \operatorname{Gau} P$, $\forall t \in \mathbb{R}$. We denote by $\operatorname{gau} P \subset \operatorname{aut} P$ the ideal of all $\pi$-vertical $G$-invariant vector fields on $P$, which is called the *gauge algebra* of $P$.

The $G$ action on $P$ lifts to an action on the tangent bundle $TP$ defined by $v_u \cdot g := T_u R_g(v_u)$, $\forall v_u \in T_u P$, $\forall g \in G$. Denote by $[v_u]_G$ the equivalence class defined by $v_u$. The quotient $(TP)/G$ is a differentiable manifold and it is endowed with a vector bundle structure $[v_u]_G \in (TP)/G \mapsto \pi(u) \in M$ over $M$ (see [1]), whose global sections can be naturally identified with $\operatorname{aut} P$; i.e., $\operatorname{aut} P \simeq \Gamma(M, (TP)/G)$. This isomorphism is given in the following way. If $X \in \operatorname{aut} P$, then $s_X \in \Gamma(M, (TP)/G)$ is given by $s_X(x) := [X(u)]_G \in (TP)/G$, where $\pi(u) = x$. Conversely, if $s \in \Gamma(M, (TP)/G)$, then $X_s \in \operatorname{aut} P$ is given by $X_s(u) := v_p$, where $s(x) = [v_u]_G$, with $\pi(u) = x$.

Let $\pi_{\mathfrak{g}} : \operatorname{ad} P \to M$ be the adjoint bundle of $P$, the bundle associated to $P$ by the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$, that is, $\operatorname{ad} P = (P \times \mathfrak{g})/G$ where the $G$-action on $P \times \mathfrak{g}$ is

$$(u, B) \cdot g = (u \cdot g, \operatorname{Ad}_{g^{-1}} B), \quad \forall (u, B) \in P \times \mathfrak{g},\ \forall g \in G.$$

Given a pair $(u, B) \in P \times \mathfrak{g}$ we shall denote its $G$-orbit in $\operatorname{ad} P$ by $(u, B)_{\mathrm{ad}}$. We also remark that the fibers $(\operatorname{ad} P)_x$ are endowed with a Lie algebra structure uniquely determined by the condition

$$\big[(u, A)_{\mathrm{ad}}, (u, B)_{\mathrm{ad}}\big] = \big(u, [A, B]\big)_{\mathrm{ad}}, \quad \forall u \in \pi^{-1}(x),\ \forall A, B \in \mathfrak{g}, \tag{2.1}$$

where $[\,,\,]$ is the bracket in $\mathfrak{g}$.

The gauge algebra of $P$ can be identified with the sections of the adjoint bundle $\pi_{\mathfrak{g}}$ (cf. [11, III.35], [13, I.Proposition 5.4]), that is, $\operatorname{gau} P \simeq \Gamma(M, \operatorname{ad} P)$.

We obtain an exact sequence of vector bundles over $M$ (the so-called Atiyah sequence [1, Theorem 1]),

$$0 \to \operatorname{ad} P \to (TP)/G \xrightarrow{[T\pi]_G} TM \to 0, \tag{2.2}$$

where the first map is given by

$$(u, B)_{\mathrm{ad}} \mapsto [B^*_u]_G, \quad \forall (u, B)_{\mathrm{ad}} \in \operatorname{ad} P,$$

$B^*$ denotes the infinitesimal generator (or fundamental vector field) of the action given by $B \in \mathfrak{g}$, and $[T\pi]_G([v_u]_G) := T_u\pi(v_u)$.

### *2.3. The bundle of connections*

connections on $P$ and splittings of the Atiyah sequence. Accordingly, one defines the bundle of connections $p : C := C(P) \to M$ as the subbundle of $\mathrm{Hom}(TM, (TP)/G)$ determined fiberwise by all $\mathbb{R}$-linear mappings $\lambda : T_xM \to ((TP)/G)_x$ such that $[T\pi]_G \circ \lambda = 1_{T_xM}$ (see, e.g., [7, Definition 4.5], [8,9], [14, §52]).

The set of connections on $P$ can thus be identified with the set of global sections of $p : C \to M$. Let $\sigma_\Gamma : M \to C$ be the section of $C$ induced by $\Gamma$. An element $\lambda : T_xM \to ((TP)/G)_x$ of the fiber of $C$ over a point $x \in M$ is nothing but a 'connection at a point $x$'; i.e., $\lambda$ induces by horizontal lift or, equivalently, a complementary subspace $H_u^\Gamma$ of the vertical subspace $V_u(P) \subset T_u(P)$ for all $u \in \pi^{-1}(x)$. If a linear mapping $h : T_xM \to (\mathrm{ad}\, P)_x$ is added to $\lambda$ we obtain another element $\lambda' = h + \lambda \in C$, as $h \in \ker[T\pi]_G$. In this way $C$ is an affine bundle modeled over the vector bundle $\mathrm{Hom}(TM, \mathrm{ad}\, P) \simeq T^*M \otimes \mathrm{ad}\, P$.

### 2.3.2. *Coordinates on C*

If $\pi : P \to M$ is trivial over an open subset $U \subseteq M$, that is, $\pi^{-1}(U) \cong U \times G$, then for every $B \in \mathfrak{g}$ we can define a one-parameter group of gauge transformations over $U$ by setting $\varphi_t^B(x, g) = (x, \exp(tB) \cdot g)$, $x \in U$. Denote by $\widetilde{B}$ the corresponding infinitesimal generator. Remark that $\widetilde{B}$ is a vertical $G$-invariant vector field on $U \times G$, that is, $\widetilde{B} \in \mathrm{gau}(U \times G)$. Set $\dim G = m$ and $\dim M = n$. Let $\{B_1, \ldots, B_m\}$ be a basis of $\mathfrak{g}$. Then, $\widetilde{B}_1, \ldots, \widetilde{B}_m$ is a basis of sections for $\mathrm{ad}\, \pi^{-1}(U)$. Let $\Gamma$ be a connection on $P$. Since $\sigma_\Gamma : TU \to T(\pi^{-1}(U))/G$ is a section of $[T\pi]_G$ in (2.2), there exist unique functions $A_j^\alpha(\Gamma) \in C^\infty(U)$ such that,

$$\sigma_\Gamma\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j} - A_j^\alpha(\Gamma)\widetilde{B}_\alpha, \quad 1 \leqslant j \leqslant n. \tag{2.4}$$

The functions $(x^j; A_j^\alpha)$, $1 \leqslant j \leqslant n$, $1 \leqslant \alpha \leqslant m$, induce a coordinate system on $p^{-1}(U) = C(\pi^{-1}(U)) = C|U$. Note that $\dim C = n(m + 1)$.

### 2.3.3. *The homomorphism* $\mathrm{aut}\, P \to \mathfrak{X}(C)$

As is well known (cf. [13, II.Proposition 6.2(b)]), each $\Phi \in \mathrm{Aut}\, P$ acts on the connections of $P$ by pushing forward horizontal distributions; i.e., for every connection $\Gamma$ with horizontal distribution $H_u^\Gamma$, $u \in P$, the image $\Gamma' = \Phi(\Gamma)$ is the connection corresponding to the distribution

$$H_{\Phi(u)}^{\Gamma'} = T_u\Phi\big(H_u^\Gamma\big), \quad u \in P. \tag{2.5}$$

In fact, if $\omega_\Gamma$ is the connection form defined by $\Gamma$, we have $\omega_{\Gamma'} = (\Phi^{-1})^*\omega_\Gamma$. If $\Psi \in \mathrm{Aut}\, P$ is another automorphism, then $(\Psi \circ \Phi)(\Gamma) = \Psi(\Phi(\Gamma))$. For each $\Phi \in \mathrm{Aut}\, P$ there exists a unique diffeomorphism

$$\Phi_C : C \to C \tag{2.6}$$

such that $p \circ \Phi_C = \phi \circ p$, where $\phi : M \to M$ is the diffeomorphism induced from $\Phi$, and $\Phi_C \circ \sigma_\Gamma = \sigma_{\Phi(\Gamma)} \circ \phi$, for every connection $\Gamma$ on $P$. The map $\Phi \in \mathrm{Aut}\, P \mapsto \Phi_C \in \mathrm{Diff}\, C$ is therefore a group homomorphism. If $\Phi_t$ is the flow of a $G$-invariant vector field $X \in \mathrm{aut}\, P$, then $(\Phi_t)_C$ is a flow on $C$ and the corresponding infinitesimal generator is denoted by $X_C$. This defines the Lie algebra homomorphism

$$\mathrm{aut}\, P \to \mathfrak{X}(C), \quad X \mapsto X_C. \tag{2.7}$$

The vector field $X_C$ is projectable onto $X'$, which equals the projection of $X$ onto $M$. By using a coordinate chart $(U; x^1, \ldots, x^n)$ on $M$ and the basis of sections $\widetilde{B}_1, \ldots, \widetilde{B}_m$ of $\mathrm{ad}\, \pi^{-1}(U)$ introduced above, it follows that each $X \in \operatorname{aut}\pi^{-1}(U)$ can be written as

$$X = f^i \frac{\partial}{\partial x^i} + g^\alpha \widetilde{B}_\alpha, \quad f^i, g^\alpha \in C^\infty(U),\ 1 \leqslant i \leqslant n;\ 1 \leqslant \alpha \leqslant m. \tag{2.8}$$

Using the definition of the homomorphism (2.7), a simple calculation shows that (see, for example, [9])

$$X_C = f^i \frac{\partial}{\partial x^i} - \left( \frac{\partial g^\alpha}{\partial x^i} - c^\alpha_{\beta\gamma} g^\beta A^\gamma_i + \frac{\partial f^h}{\partial x^i} A^\alpha_h \right) \frac{\partial}{\partial A^\alpha_i}, \tag{2.9}$$

where $c^\alpha_{\beta\gamma}$ are the structure constants of $\mathfrak{g}$, that is, $[B_\beta, B_\gamma] = c^\alpha_{\beta\gamma} B_\alpha$.

## 2.4. *Jets and contact transformations*

Given a fiber bundle $p : E \to M$, denote by $p_{10} : J^1E \to E$ the first jet bundle of $p$, that is, the bundle of 1-jets $j^1_x s$ of local sections $s : U \subset M \to E$, $x \in U$, of the projection $p$. The maps $p_{10}(j^1_x s) = s(x)$ and $p_1 : J^1E \to M$, $p(j^1_x s) = x$, are called the *target* and the *source projections* (see, for example, [19]). Let $(x^i, y^a)$ be an adapted coordinate system on $E$. The induced coordinate system $(x^i, y^a, y^a_{,i})$ on $J^1E$ is defined by $y^a_{,i}(j^1_x s) = (\partial(y^a \circ s))/\partial x^i)(x)$, for any local section $s$ of $E$.

Let $(\Psi, \psi)$, $\Psi : E \to E$, $\psi : M \to M$, $\psi \circ p = p \circ \Psi$, be a fibered diffeomorphism of the bundle $p : E \to M$. Denote by $\Psi^{(1)} \in \operatorname{Diff}(J^1E)$ its 1-jet prolongation, defined by

$$\Psi^{(1)}\big(j^1_x s\big) = j^1_{\psi(x)}\big(\Psi \circ s \circ \psi^{-1}\big), \quad j^1_x s \in J^1E.$$

The mapping $\Psi \mapsto \Psi^{(1)}$, is a monomorphism from the group $\operatorname{Diff}_M(E)$ of fibered diffeomorphisms of $E$ to the group $\operatorname{Diff}(J^1E)$. Its infinitesimal version gives rise to a monomorphism of Lie algebras

$$\mathfrak{X}_M(E) \to \mathfrak{X}\big(J^1E\big), \quad Y \mapsto Y^{(1)},$$

where $Y^{(1)}$ is the infinitesimal generator of $\Psi^{(1)}_t$, $\Psi_t$ is the flow of $Y$, and $\mathfrak{X}_M(E)$ denotes the Lie algebra of $p$-projectable vector fields on $E$. If $Y = f^i \partial/\partial x^i + h^a \partial/\partial y^a$, $f^i \in C^\infty(M)$, $h^a \in C^\infty(E)$, is the local expression of a $p$-projectable vector field, then the expression of $Y^{(1)}$ is (see, for example, [19, §4.4])

$$Y^{(1)} = f^i \frac{\partial}{\partial x^i} + h^a \frac{\partial}{\partial y^a} + \left( \frac{\partial h^a}{\partial x^i} - y^a_{,j} \frac{\partial f^j}{\partial x^i} \right) \frac{\partial}{\partial y^a_{,i}}.$$

In particular, we are concerned with the first jet bundle $p_{10} : J^1C \to C$ of the bundle of connections $p : C \to M$. In this case, if (2.8) is the local expression of an infinitesimal automorphism $X \in \operatorname{aut} P$, from formula (2.9) of the local representation of $X_C \in \mathfrak{X}(C)$, we have

$$\begin{aligned} X_C^{(1)} = {} & f^i \frac{\partial}{\partial x^i} - \left( \frac{\partial g^\alpha}{\partial x^i} - c^\alpha_{\beta\gamma} g^\beta A^\gamma_i + \frac{\partial f^h}{\partial x^i} A^\alpha_h \right) \frac{\partial}{\partial A^\alpha_i} \\ & - \left( \frac{\partial^2 g^\alpha}{\partial x^i \partial x^j} - c^\alpha_{\beta\gamma} \frac{\partial g^\beta}{\partial x^j} A^\gamma_i - c^\alpha_{\beta\gamma} g^\beta A^\gamma_{i,j} + \frac{\partial^2 f^h}{\partial x^i \partial x^j} A^\alpha_h + \frac{\partial f^h}{\partial x^i} A^\alpha_{h,j} \right) \frac{\partial}{\partial A^\alpha_{i,j}}. \end{aligned} \tag{2.10}$$

### 2.5. Lie derivative with respect to a multi-vector field

Recall (cf. [4, p. 79]) that a decomposable multi-vector field $\chi = X_1 \wedge \cdots \wedge X_d \in \bigwedge^d \mathfrak{X}(M)$, induces a $C^\infty(M)$-linear graded endomorphism $i_\chi : \Omega^\bullet(M) = \bigoplus_{d\in\mathbb{Z}} \Omega^d(M) \to \Omega^\bullet(M)$ of degree $-d$ by setting

$$i_\chi \omega_r = (i_{X_1} \circ \cdots \circ i_{X_k})\omega_r \in \Omega^{r-k}(M), \quad \omega_r \in \Omega^r(M), \tag{2.11}$$

where $i_X$ denotes the usual interior product or contraction on the first index. For an arbitrary multi-vector field $\chi$ define $i_\chi$ by extending the formula above by linearity. Then the Lie derivative $L_\chi : \Omega^\bullet(M) \to \Omega^\bullet(M)$ with respect to $\chi \in \bigwedge^d \mathfrak{X}(M)$ is defined by

$$L_\chi = i_\chi \circ \mathrm{d} - (-1)^d \mathrm{d} \circ i_\chi = [i_\chi, \mathrm{d}]. \tag{2.12}$$

Note that $L_\chi$ is a graded operator of degree $-d+1$.

More generally, let $G$ be a Lie group and let $\bigwedge^\bullet \mathfrak{X}(M) \otimes S^\bullet(\mathfrak{g}^*)$ be the space of $S^\bullet(\mathfrak{g}^*)$-valued multi-vector fields, $S^\bullet(\mathfrak{g}^*)$ being the symmetric algebra of $\mathfrak{g}^*$. Let $\mathcal{X} = \chi \otimes f$ be an element of $\bigwedge^k \mathfrak{X}(M) \otimes S^l(\mathfrak{g}^*)$. Define

$$L_{\mathcal{X}} : \Omega^r(M) \to \Omega^{r-d+1}(M) \otimes S^\bullet(\mathfrak{g}^*)$$

by the condition $L_{\mathcal{X}}(\omega_r) = L_\chi \omega_r \otimes f$, letting the polynomial act trivially. For an arbitrary element $\mathcal{X} \in \bigwedge^d \mathfrak{X}(M) \otimes S^l(\mathfrak{g}^*)$, the operator $L_{\mathcal{X}}$ is defined by linear extension.

## 3. Invariant Lagrangians

Given a fibered manifold $p : E \to M$, a Lagrangian density is a fibered mapping over $M$,

$$\Lambda : J^1(E) \to \bigwedge^n T^*M,$$

with $n = \dim M$. If the base manifold is connected and oriented by a volume form $v$, then every Lagrangian density can be written as $\Lambda = \mathcal{L} v$ for a certain mapping $\mathcal{L} : J^1(E) \to \mathbb{R}$, called the Lagrangian.

If $M$ is compact, the action functional is defined to be

$$\pounds : \Gamma(E) \to \mathbb{R}, \quad \pounds(s) = \int_M \Lambda\big(j^1 s\big).$$

A Lagrangian $\mathcal{L} : J^1(E) \to \mathbb{R}$ is said to be *variationally trivial* if the Euler–Lagrange equations hold for every local section of the bundle $p : E \to M$; that is, every local section is a critical point of the action functional defined by $\mathcal{L}$.

**Proposition 1** (e.g., see [15,16]). *Given a fibered manifold $E \to M$, a Lagrangian $\mathcal{L} : J^1 E \to \mathbb{R}$ is variationally trivial if and only if the following systems of differential equations are satisfied*:

$$\frac{\partial^2 \mathcal{L}}{\partial y^a_{,i} \partial y^b_{,j}} + \frac{\partial^2 \mathcal{L}}{\partial y^a_{,j} \partial y^b_{,i}} = 0, \quad i, j = 1, \ldots, n;\ a, b = 1, \ldots, m,$$

$$\frac{\partial \mathcal{L}}{\partial y^a} = \frac{\partial^2 \mathcal{L}}{\partial x^i \partial y^a_{,i}} + \frac{\partial^2 \mathcal{L}}{\partial y^b \partial y^a_{,i}} y^b_{,i}, \quad a = 1, \ldots, m,$$

*where* $(x^i, y^a, y^a_{,i})$ *is the induced coordinate system on* $J^1E$ *by an adapted coordinate system* $(x^i, y^a)$ *on* $E$ (*see Section* 2.4).

Two Lagrangians $\mathcal{L}$, $\mathcal{L}'$ are said to be *variationally equivalent* if they define the same variational problem, that is, $\mathcal{L} - \mathcal{L}'$ is variationally trivial. Therefore, the characterization of variational triviality solves the problem of variational equivalence.

### 3.1. Gauge invariance

A Lagrangian density $\Lambda = \mathcal{L}v$, $\mathcal{L}: J^1C \to \mathbb{R}$, on the bundle of connections is said to be *gauge invariant* (also called *natural* by some authors; e.g., [14, XII]) if the directional derivative of $\mathcal{L}$ in the direction of $X_C^{(1)}$ vanishes, that is,

$$X_C^{(1)}[\mathcal{L}] = 0,$$

for every $X \in \operatorname{gau} P$, where $X_C^{(1)}$ denotes the 1-jet prolongation of the representation $\operatorname{aut} P \to \mathfrak{X}(C)$ (see Section 2.4). Similarly, a Lagrangian density is said to be aut $P$-*invariant* if

$$L_{X_C^{(1)}}(\mathcal{L}p_1^*v) = X_C^{(1)}[\mathcal{L}]p_1^*v + \mathcal{L}L_{X_C^{(1)}}(p_1^*v) = 0, \quad \forall X \in \operatorname{aut} P,$$

where $p_1: J^1C \to M$ is the source projection. The vector field $X_C^{(1)}$ is $p_1$-projectable to $X'$; the vector field $X'$ is $\pi$-related to $X$. Then, the condition of aut $P$-invariance yields

$$X_C^{(1)}[\mathcal{L}] + \mathcal{L}\operatorname{div} X' = 0, \quad \forall X \in \operatorname{aut} P. \tag{3.1}$$

If $X \in \operatorname{gau} P$, then $X' = 0$ and the definition of gauge invariance is recovered.

A classical result by Utiyama (see [3,7,9,20]) gives a full characterization of gauge invariant Lagrangians. From a geometrical point of view, this characterization has the following description.

Given a principal $G$-bundle $\pi : P \to M$, the curvature mapping

$$\Omega : J^1C \to \bigwedge^2 T^*M \otimes \operatorname{ad} P,$$

is defined by the condition $\Omega(j^1_x\sigma_\Gamma) = (\Omega_\Gamma)_x$, $\Omega_\Gamma$ being the curvature 2-form of the connection $\sigma_\Gamma$, regarded as a 2-form on $M$ with values in the adjoint bundle ad $P$. Consider the coordinate system $(x^i, A^\alpha_i, A^\alpha_{i,j})$ on $J^1C$ given in Section 2.4 and the naturally induced coordinate system $(x^i, R^\alpha_{ij})$, $1 \leqslant i < j \leqslant n$, $1 \leqslant \alpha \leqslant m$, on $\bigwedge^2 T^*M \otimes (\operatorname{ad} P)$ defined by setting

$$\omega_2 = \sum_{i<j} R^\alpha_{ij}(\omega_2)\big(\mathrm{d}x^i\big)_x \wedge \big(\mathrm{d}x^j\big)_x \otimes (\widetilde{B}_\alpha)_x, \quad \omega_2 \in \bigwedge^2 T^*_xM \otimes (\operatorname{ad} P)_x. \tag{3.2}$$

In these coordinates, the local expression of the curvature mapping is

$$R^\alpha_{ij}(\Omega) = A^\alpha_{i,j} - A^\alpha_{j,i} - c^\alpha_{\beta\gamma}A^\beta_i A^\gamma_j. \tag{3.3}$$

The Utiyama Theorem states that a Lagrangian mapping $\mathcal{L}: J^1C \to \mathbb{R}$ is gauge invariant if and only if it factors through the curvature by means of an adjoint invariant function. More precisely, $\mathcal{L}$ is gauge invariant if and only if $\mathcal{L} = \overline{\mathcal{L}} \circ \Omega$, for a function $\overline{\mathcal{L}}: \bigwedge^2 T^*M \otimes \operatorname{ad} P \to \mathbb{R}$ invariant under the adjoint representation of the Lie algebra $\mathfrak{g}$ in each fiber $(\operatorname{ad} P)_x$, $x \in M$.

We remark that the action functional of an $\operatorname{aut} P$-invariant density $\mathcal{L}v$ is also invariant under automorphisms, but the converse is not true. For example, the action functional of the Chern–Simons density is $\operatorname{aut} P$-invariant, but the Lagrangian itself is not, even in the Abelian case.

### 3.2. Lagrangian associated to a Weil polynomial

Given a Lie group $G$ and its Lie algebra $\mathfrak{g}$, we denote by $k\mathfrak{g} = \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$ the direct sum of $k$ copies of $\mathfrak{g}$. Similarly, given a vector bundle, we denote by $k \operatorname{ad} P = \operatorname{ad} P \oplus \cdots \oplus \operatorname{ad} P$ the Whitney sum of $k$ copies of the associated adjoint bundle.

A symmetric multilinear map $f : k\mathfrak{g} \to \mathbb{R}$ is called a *Weil polynomial* of degree $k$ if it is invariant under the adjoint representation of $G$ on $\mathfrak{g}$; i.e., if

$$f(\operatorname{Ad}_g B_1, \dots, \operatorname{Ad}_g B_k) = f(B_1, \dots, B_k), \quad \forall B_1, \dots, B_k \in \mathfrak{g},\ \forall g \in G.$$

The set of all Weil polynomials of degree $k$ is denoted by $I_k^G$. The space $I^G = \bigoplus_{k \geqslant 0} I_k^G$ is a subalgebra of the symmetric algebra $S^\bullet(\mathfrak{g}^*)$ of the dual $\mathfrak{g}^*$ (cf. [13, XII.§1]).

Given $f \in I_k^G$, we define the function

$$\bar{f} : k \operatorname{ad} P \to \mathbb{R},$$

by the condition

$$\bar{f}\big((u, B_1)_{\mathrm{ad}}, \dots, (u, B_k)_{\mathrm{ad}}\big) = f(B_1, \dots, B_k), \tag{3.4}$$

with $(u, B_i)_{\mathrm{ad}} \in (\operatorname{ad} P)_x$, $1 \leqslant i \leqslant k$, $x \in M$, $u \in \pi^{-1}(x)$. Taking into account that $(u \cdot g, \operatorname{Ad}_{g^{-1}} B)_{\mathrm{ad}} = (u, B)_{\mathrm{ad}}$, the invariance condition of $f$ under the adjoint representation allows one to see that the previous definition makes sense, that is, it does not depend on the choice of the point $u \in \pi^{-1}(x)$.

Let $\pi : P \to M$ be a principal $G$-bundle over a manifold $M$ of even dimension $\dim M = n = 2k$ and let $f \in I_k^G$ be a Weil polynomial of degree $k$. We define the fibered mapping

$$\overline{\Lambda}_f : \bigwedge^2 T^*M \otimes \operatorname{ad} P \to \bigwedge^n T^*M, \tag{3.5}$$

by

$$\overline{\Lambda}_f(w)(X_1, \dots, X_n) = \frac{1}{(2k)!} \sum_{\tau \in S_{2k}} \varepsilon(\tau) \bar{f}\big(w(X_{\tau(1)}, X_{\tau(2)}), \dots, w(X_{\tau(2k-1)}, X_{\tau(2k)})\big),$$

for every $w \in (\bigwedge^2 T^*M \otimes \operatorname{ad} P)_x$, and arbitrary vectors $X_1, \dots, X_n \in T_xM$, $x \in M$, where $\varepsilon(\tau)$ is the signature of the permutation $\tau$. If $M$ is endowed with a fixed volume form $v$ we define $\overline{\mathcal{L}}_f \in C^\infty(\bigwedge^2 T^*M \otimes \operatorname{ad} P)$ as the unique function satisfying the condition $\overline{\Lambda}_f = \overline{\mathcal{L}}_f v$. Each Weil polynomial thus defines a Lagrangian density

$$\Lambda_f = \overline{\Lambda}_f \circ \Omega : J^1(CP) \to \bigwedge^n T^*M,$$

by composing $\overline{\Lambda}_f$ with the curvature mapping $\Omega$.

**Proposition 2.** *Given a Weil polynomial $f \in I_k^G$, the Lagrangian density $\Lambda_f$ defined above is* $\operatorname{aut} P$-*invariant.*

**Proof.** The mapping $\overline{\Lambda}_f : \bigwedge^2 T^*M \otimes \operatorname{ad} P \to \bigwedge^n T^*M$, defined in (3.5) is invariant under the adjoint representation on $\operatorname{ad} P$ because of the invariance condition of the Weil polynomial $f$. Then, by virtue of the Utiyama Theorem, the Lagrangian density $\Lambda_f = \overline{\Lambda}_f \circ \Omega$ is gauge invariant. For the invariance under the full algebra of infinitesimal automorphisms, as the problem is local, we only need to consider the trivial bundle $P = M \times G$. We only have to check condition (3.1) for vector fields of the type $X = f^i(\partial/\partial x^i)$. Let $\{\varphi_t\}$ be the flow of $X$ and $\Phi : P \to P$, $\Phi(x,g) = (\varphi(x), g)$, where for the sake of simplicity, we have dropped the variable $t$. For any $j^1_x\sigma_\Gamma \in J^1C$, we have

$$\begin{aligned}\big(\big(\Phi_C^{(1)}\big)^*\Lambda_f\big)\big(j^1_x\sigma_\Gamma\big) &= \varphi^*\big(\Lambda_f\big(\Phi_C^{(1)}\big(j^1_x\sigma_\Gamma\big)\big)\big) = \varphi^*\big(\Lambda_f\big(j^1_{\varphi(x)}\sigma_{\Phi(\Gamma)}\big)\big)\\ &= \varphi^*\overline{\Lambda}_f\big(\Omega\big(j^1_{\varphi(x)}\sigma_{\Phi(\Gamma)}\big)\big) \overset{(1)}{=} \varphi^*\overline{\Lambda}_f\big(\big(\varphi^{-1}\big)^*\Omega\big(j^1_x\sigma_\Gamma\big)\big)\\ &\overset{(2)}{=} \varphi^*\big(\varphi^{-1}\big)^*\overline{\Lambda}_f\big(\Omega\big(j^1_x\sigma_\Gamma\big)\big) = \Lambda_f\big(j^1_x\sigma_\Gamma\big),\end{aligned}$$

where the equality (1) is a general property of the curvature of a connection under a transformation $\varphi \in \operatorname{Diff}(M)$ and equality (2) is a direct consequence of the definition of $\overline{\Lambda}_f$ given above. The proof is complete by taking into account the formula of the Lie derivative

$$L_{X_C^{(1)}}\Lambda_f = \left.\frac{d}{dt}\right|_{t=0}\big((\Phi_t)_C^{(1)}\big)^*\Lambda_f. \qquad \square$$

### 3.3. *Characterization of* aut $P$*-invariance*

**Lemma 1.** *Let $N$ be a manifold and let $f \in C^\infty(\mathbb{R}^n \times N)$ be a map such that*

$$f = x^i\frac{\partial f}{\partial x^i}, \tag{3.6}$$

*$(x^i)$ being the standard coordinates in $\mathbb{R}^n$. Then the functions $\partial f/\partial x^i$, $1 \leqslant i \leqslant n$, are independent of $(x^i)$; i.e., $\partial f/\partial x^i \in C^\infty(N)$.*

**Proof.** Let $(x,y) \in \mathbb{R}^n \times N$ and let $\phi(t) = f(tx,y)$. Condition (3.6) gives $\phi(0) = 0$, and by the chain rule,

$$\frac{d\phi}{dt} = x^i\frac{\partial f}{\partial x^i}(tx,y) = \frac{1}{t}\phi(t).$$

The solution to this differential equation is $\phi(t) = \phi(1)t = f(x,y)t$, so that

$$t\cdot f(x,y) = \phi(t) = f(tx,y) = \sum_{i=1}^n tx^i\big(\partial f/\partial x^i\big)(tx,y).$$

Hence, $f(x,y) = \sum_{i=1}^n x^i(\partial f/\partial x^i)(tx,y)$ and letting $t = 0$, we obtain

$$f(x,y) = x^i\frac{\partial f}{\partial x^i}(0,y),$$

thus finishing the proof. $\square$

**Lemma 2.** *Let $\mathfrak{g}$ be a Lie algebra and let $(x^i, R^\alpha_{ij})$, $1 \leqslant i < j \leqslant n$, $\alpha = 1, \dots, m$, be the coordinate system on $\bigwedge^2 T^*\mathbb{R}^n \otimes \mathfrak{g}$ defined in* (3.2). *If $f \in C^\infty(\bigwedge^2 T^*\mathbb{R}^n \otimes \mathfrak{g})$ is such that*

$$f = \sum_{j,\alpha} R^\alpha_{ij} \frac{\partial f}{\partial R^\alpha_{ij}}, \quad i = 1, \dots, n, \tag{3.7}$$

*where $R^\alpha_{ij} = -R^\alpha_{ji}$ whenever $i \geqslant j$, then*

(1) *for $n$ odd, $f$ is the zero function*;
(2) *for $n$ even, say $n = 2k$, we have*

$$f = \sum_{\tau \in S_n} \lambda^\tau_{\alpha_1, \dots, \alpha_k} R^{\alpha_1}_{\tau(1)\tau(2)} \dots R^{\alpha_k}_{\tau(n-1)\tau(n)},$$

*where $S_n$ denotes the permutation group and $\lambda^\tau_{\alpha_1, \dots, \alpha_k} \in C^\infty(\mathbb{R}^n)$.*

**Proof.** By induction on $n$. For $n = 2$, we have

$$f = R^\alpha_{12} \frac{\partial f}{\partial R^\alpha_{12}},$$

and by virtue of Lemma 1, $\partial f / \partial R^\alpha_{12}$, $\alpha = 1, \dots, m$, are in $C^\infty(\mathbb{R}^n)$. For $n = 3$ and letting $i = 1$ in formula (3.7), we have

$$f = R^\alpha_{12} \frac{\partial f}{\partial R^\alpha_{12}} + R^\alpha_{13} \frac{\partial f}{\partial R^\alpha_{13}}. \tag{3.8}$$

In this case, for any $\alpha = 1, \dots, n$, Lemma 1 implies that $\partial f / \partial R^\alpha_{12}$ does not depend on $R^1_{12}, \dots, R^m_{12}$, $R^1_{13}, \dots, R^m_{13}$. Setting $i = 2$ in Eq. (3.7) yields

$$f = R^\alpha_{21} \frac{\partial f}{\partial R^\alpha_{21}} + R^\alpha_{23} \frac{\partial f}{\partial R^\alpha_{23}},$$

and then (by Lemma 1), $\partial f / \partial R^\alpha_{21} = -\partial f / \partial R^\alpha_{12}$ does not depend on $R^1_{23}, \dots, R^m_{23}$; that is, $\partial f / \partial R^\alpha_{12} \in C^\infty(\mathbb{R}^n)$, $\alpha = 1, \dots, n$. The same conclusion is obtained for $\partial f / \partial R^\alpha_{13}$ and $\partial f / \partial R^\alpha_{23}$. Now, taking derivatives in (3.8) with respect to $R^\alpha_{23}$ we have $\partial f / \partial R^\alpha_{23} = 0$, $\alpha = 1, \dots, n$. We proceed similarly with $\partial f / \partial R^\alpha_{12}$ and $\partial f / \partial R^\alpha_{13}$. Thus $f = 0$. For an arbitrary $n \geqslant 4$, we define $f^{12}_\alpha = \partial f / \partial R^\alpha_{12}$, $\alpha = 1, \dots, m$. From Eq. (3.7) for $i = 1$ and Lemma 1, we conclude that $f^{12}_\alpha$ does not depend on $R^\beta_{1k}$, $k = 1, \dots, n$, $\beta = 1, \dots, m$, and it does not depend on $R^\beta_{2k}$ either, as $f^{12}_\alpha = -f^{21}_\alpha$. Hence $f^{12}_\alpha$ can be seen as a function on $\bigwedge^2 T^*\mathbb{R}^{n-2} \otimes \mathfrak{g}$, where $(x^3, \dots, x^n)$ are the variables in $\mathbb{R}^{n-2}$. By applying $\partial / \partial R^\alpha_{12}$ to (3.7) for $i > 2$, we obtain

$$f^{12}_\alpha = \sum_{j > 2, \beta} R^\beta_{ij} \frac{\partial f^{12}_\alpha}{\partial R^\beta_{ij}}, \quad i > 2,$$

and therefore $f^{12}_\alpha$ satisfies the conditions of the statement for $n - 2$. Hence, if $n - 2$ (and also $n$) is odd, then by the induction hypothesis $f^{12}_\alpha = 0$, $\alpha = 1, \dots, m$. Similarly, for any other indices we have $f^{ij}_\alpha = \partial f / \partial R^\alpha_{ij} = 0$, $i, j = 1, \dots, n$, $\alpha = 1, \dots, m$, and we conclude $f = 0$. For $n$ even, the induction hypothesis implies

$$f_\alpha^{12} = \frac{\partial f}{\partial R_{12}^\alpha} = \sum_{\tau\in S_{n-2}} \lambda^\tau_{\alpha,\alpha_2,\dots,\alpha_k} R^{\alpha_2}_{\tau(2)\tau(4)} \dots R^{\alpha_k}_{\tau(n-1)\tau(n)},$$

and similarly for $f_\alpha^{1j}$, $j = 3, \dots, n$, $\alpha = 1, \dots, m$. With these expressions, Eq. (3.7) completes the proof. □

**Theorem 1.** *Let $\pi : P \to M$ be a principal $G$-bundle with $G$ connected and $M$ connected and orientable. Then*:

(1) *If* $\dim M$ *is odd, the only* $\operatorname{aut} P$*-invariant Lagrangian density is the zero density.*
(2) *If* $\dim M$ *is even* (*say* $\dim M = 2k$), *all* $\operatorname{aut} P$*-invariant Lagrangian densities are of the form $\Lambda_f$, with $f \in I_k^G$* (*see Section* 3.2).

**Proof.** As $M$ is orientable, fix in what follows a volume form $v$ on $M$. Let $\mathcal{L}v$ be an $\operatorname{aut} P$-invariant Lagrangian density. In particular, $\mathcal{L}v$ is gauge invariant and by virtue of Utiyama's Theorem, there exists a function $\overline{\mathcal{L}} : \bigwedge^2 T^*M \otimes \operatorname{ad} P \to \mathbb{R}$, invariant under the adjoint representation, such that

$$\mathcal{L} = \overline{\mathcal{L}} \circ \Omega,$$

$\Omega$ being the curvature mapping. Let $(x^1, \dots, x^n)$ be a coordinate system in $M$ such that $v = dx^1 \wedge \dots \wedge dx^n$ and let $(x^i, A_i^\alpha; A_{i,j}^\alpha)$, $i, j = 1, \dots, n$, $\alpha = 1, \dots, m$, and $(x^i, R_{ij}^\alpha)$, $1 \leqslant i < j \leqslant n$, $\alpha = 1, \dots, m$, be the induced systems of coordinates defined on $J^1C$ and $\bigwedge^2 T^*M \otimes \operatorname{ad} P$ respectively. We take an infinitesimal automorphism $X \in \operatorname{aut} P$ of the form $X = f^j \partial/\partial x^j$. Its lifting to $J^1C$ (cf. formula (2.10)) is

$$X_C^{(1)} = f^j \frac{\partial}{\partial x^j} - \frac{\partial f^i}{\partial x^j} A_i^\alpha \frac{\partial}{\partial A_j^\alpha} - \left( \frac{\partial f^l}{\partial x^j} A^\alpha_{l,i} + \frac{\partial f^l}{\partial x^i} A^\alpha_{j,l} + \frac{\partial^2 f^l}{\partial x^j \partial x^i} A_l^\alpha \right) \frac{\partial}{\partial A^\alpha_{j,i}}.$$

The condition of $\operatorname{aut} P$-invariance $X_C^{(1)}[\mathcal{L}] + \mathcal{L} \operatorname{div} X' = 0$ for this vector field is

$$f^j \frac{\partial \mathcal{L}}{\partial x^j} - \frac{\partial f^i}{\partial x^j} A_i^\alpha \frac{\partial \mathcal{L}}{\partial A_j^\alpha} - \left( \frac{\partial f^l}{\partial x^j} A^\alpha_{l,i} + \frac{\partial f^l}{\partial x^i} A^\alpha_{j,l} + \frac{\partial^2 f^l}{\partial x^j \partial x^i} A_l^\alpha \right) \frac{\partial \mathcal{L}}{\partial A^\alpha_{j,i}} + \mathcal{L} \frac{\partial f^j}{\partial x^j} = 0. \tag{3.9}$$

For $f^j \equiv 1$, $j = 1, \dots, n$, we have

$$\frac{\partial \mathcal{L}}{\partial x^j} = 0, \quad \forall j = 1, \dots, n. \tag{3.10}$$

Taking an arbitrary point $x_0 = (x_0^1, \dots, x_0^n) \in M$ and a pair of indices $i, h \in \{1, \dots, n\}$, for $f^j = (x^h - x_0^h)\delta_i^j$, $j = 1, \dots, n$, we obtain

$$\mathcal{L}\delta_i^h = A_i^\alpha \frac{\partial \mathcal{L}}{\partial A_h^\alpha} + A^\alpha_{i,l} \frac{\partial \mathcal{L}}{\partial A^\alpha_{h,l}} + A^\alpha_{l,i} \frac{\partial \mathcal{L}}{\partial A^\alpha_{l,h}}, \quad \forall h, i = 1, \dots, n, \tag{3.11}$$

at any $j^1_{x_0}\sigma \in (J^1C)_{x_0}$. On the other hand, as $\mathcal{L} = \overline{\mathcal{L}} \circ \Omega$, using the chain rule, and taking into account the local expression of the curvature mapping $\Omega$ (Eq. (3.3)), we have

$$\frac{\partial \mathcal{L}}{\partial A^\alpha_{i,j}} = \frac{\partial \overline{\mathcal{L}}}{\partial R^\alpha_{ij}} \circ \Omega,$$

$$\frac{\partial \mathcal{L}}{\partial A_i^\alpha} = -c_{\alpha\beta}^\gamma A_l^\beta \frac{\partial \overline{\mathcal{L}}}{\partial R_{il}^\gamma} \circ \Omega,$$

where $R_{ij}^\alpha = -R_{ji}^\alpha$ whenever $i \geqslant j$. These identities, when applied to Eqs. (3.10) and (3.11), give

$$\frac{\partial \overline{\mathcal{L}}}{\partial x^i} \circ \Omega = 0, \quad \delta_h^i \overline{\mathcal{L}} \circ \Omega = R_{hl}^\alpha \frac{\partial \overline{\mathcal{L}}}{\partial R_{il}^\alpha} \circ \Omega, \quad i, h = 1, \ldots, n,$$

or in a simpler way, taking into account that $\Omega$ is a surjective submersion (cf. [3, Theorem 10.2.7]), we get

$$\frac{\partial \overline{\mathcal{L}}}{\partial x^i} = 0, \quad i = 1, \ldots, n, \tag{3.12}$$

$$\delta_h^i \overline{\mathcal{L}} = R_{hl}^\alpha \frac{\partial \overline{\mathcal{L}}}{\partial R_{il}^\alpha}, \quad i, h = 1, \ldots, n. \tag{3.13}$$

Set $h = i$ in formula (3.13) and apply Lemma 2; it follows that $\overline{\mathcal{L}} = 0$ for $n$ odd and

$$\overline{\mathcal{L}} = \sum_{\tau \in S_n} \lambda_{\alpha_1, \ldots, \alpha_k}^\tau R_{\tau(1)\tau(2)}^{\alpha_1} \ldots R_{\tau(n-1)\tau(n)}^{\alpha_k}, \tag{3.14}$$

$\lambda_{\alpha_1, \ldots, \alpha_k}^\tau \in C^\infty(M)$, for $n = 2k$. In fact, $\lambda_{\alpha_1, \ldots, \alpha_k}^\tau$ are real constants because of Eq. (3.12). Set now $h \neq i$ in formula (3.13); we obtain

$$0 = R_{hl}^\alpha \frac{\partial \overline{\mathcal{L}}}{\partial R_{il}^\alpha}, \quad h \neq i.$$

Taking derivatives with respect to $R_{ha}^\beta$ first and then with respect to $R_{hb}^\gamma$, we obtain

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial R_{ia}^\beta \partial R_{hb}^\gamma} + \frac{\partial^2 \overline{\mathcal{L}}}{\partial R_{ib}^\gamma \partial R_{ha}^\beta} + R_{hl}^\alpha \frac{\partial^3 \overline{\mathcal{L}}}{\partial R_{il}^\alpha \partial R_{ia}^\beta \partial R_{hb}^\gamma} = 0,$$

for $h \neq i$, $a, b = 1, \ldots, n$, $\beta, \gamma = 1, \ldots, m$. The last summand vanishes since expression (3.14) does not contain any repeated product $R_{il}^\alpha R_{ia}^\beta$. Thus one gets

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial R_{ia}^\beta \partial R_{hb}^\gamma} + \frac{\partial^2 \overline{\mathcal{L}}}{\partial R_{ha}^\beta \partial R_{ib}^\gamma} = 0, \tag{3.15}$$

for $h \neq i$, $a, b = 1, \ldots, n$, $\beta, \gamma = 1, \ldots, m$. Using expression (3.14) for $\overline{\mathcal{L}}$, it is easy to see that Eq. (3.15) means the following: if $\tau \in S_n$ is an arbitrary permutation and $\mu \in S_n$ is the transposition $\mu(a) = b$, then we have

$$\lambda_{\alpha_1, \ldots, \alpha_k}^\tau + \lambda_{\alpha_1, \ldots, \alpha_k}^{\mu \circ \tau} = 0.$$

As transpositions generate the full group $S_n$ of permutations, for any $\tau \in S_n$ we obtain

$$\lambda_{\alpha_1, \ldots, \alpha_k}^\tau = \varepsilon(\tau) \lambda_{\alpha_1, \ldots, \alpha_k}^I,$$

where $I \in S_n$ is the identity element of the group $S_n$. Hence

$$\overline{\mathcal{L}} = \lambda_{\alpha_1, \ldots, \alpha_k}^I \sum_{\tau \in S_n} \varepsilon(\tau) R_{\tau(1)\tau(2)}^{\alpha_1} \ldots R_{\tau(n-1)\tau(n)}^{\alpha_k}. \tag{3.16}$$

From the definition of the Lagrangian density defined by a Weil polynomial (see Section 3.2), it is straightforward to see that $\overline{\mathcal{L}} = \overline{\mathcal{L}}_f$ for

$$f = \lambda^I_{\alpha_1,\dots,\alpha_k} B^{\alpha_1} \vee \cdots \vee B^{\alpha_k},$$

where $\vee$ stands for the symmetric product in $S^\bullet(\mathfrak{g}^*)$ and $\{B^1,\dots,B^m\}$ is the dual basis to $\{B_1,\dots,B_m\}$ in $\mathfrak{g}^*$. The invariance of the polynomial $f$ under the Ad-representation of $G$ on $\mathfrak{g}$ follows directly from the invariance of $\overline{\mathcal{L}}$ under the ad-representation of $\mathfrak{g}$, taking into account that $G$ is connected. $\square$

To fix notations, it is necessary to briefly review the theory of the Weil homomorphism and the constructions of characteristic classes in terms of curvature forms (see, for example, [6], [13, XII]). Given a principal connection $\Gamma$ on $\pi : P \to M$ and a Weil polynomial $f$ of degree $d$, the form $f(\Omega_\Gamma)$ is a closed $2d$-form which projects onto the base manifold $M$ and whose cohomology class does not depend on the chosen connection $\Gamma$. For $\dim M = 2k$ and $\deg f = k$, if $\sigma_\Gamma : M \to C$ is the section of the bundle of connections associated to $\Gamma$, it is easy to check that $\Lambda_f(j^1\sigma_\Gamma) = \overline{\Lambda}_f(\Omega_\Gamma) = f(\Omega_\Gamma)$. Therefore, if $M$ is compact, the action functional

$$\pounds_f(\sigma_\Gamma) = \int_M \Lambda_f\bigl(j^1\sigma_\Gamma\bigr) = \int_M f(\Omega_\Gamma)$$

defined by $\Lambda_f$ does not depend on the chosen section $\sigma_\Gamma$, that is, $\pounds_f$ is constant and thus every section is critical. We have thus proved the following.

**Corollary 1.** *Every* aut $P$*-invariant Lagrangian density is variationally trivial.*

In fact, given an aut $P$-invariant Lagrangian $\Lambda_f$, with $f \in I^G_k$, the constant value of the action functional $\pounds_f$ is nothing but the characteristic number $c_f$ defined by $f$ (see [6,18]). Roughly speaking, the aut $P$-invariance determines the characteristic numbers of the bundle $\pi : P \to M$ for $f \in I^G_k$, if $\dim M = 2k$.

**Remark 1.** As a physical consequence of this fact we can conclude that the group Aut $P$ is too large to be the group of symmetries of a model in classical field theory as all sections are critical. Indeed, the invariance under the group Aut $P$ is never found in classical field theories and Corollary 1 can be understood as a mathematical explanation of this fact. Nevertheless, these trivial Lagrangians are of great interest in quantum field theory as their topological nature has turned out to be extremely important for the quantization of fields. Corollary 1 thus reveals a link between the aut $P$-symmetry and these topological objects. See, for example, [2,17] and references therein for a discussion concerning topological Lagrangians.

**Example 1.** Consider the group $G = U(1)$ and an arbitrary $U(1)$-principal bundle $\pi : P \to M$. This is, for example, the setting for electromagnetism. In this case $\mathfrak{g} = \mathfrak{u}(1) = \mathbb{R}$ and the algebra of Weil polynomials is generated by the identity, that is, $I^{U(1)} = \mathbb{R}[t]$. For this group, the statement of Theorem 1 can be rephrased by saying that, for $\dim M$ even, every aut $P$-invariant Lagrangians is a multiple of the Pfaffian (see [5] for a complete proof).

**Example 2.** For the group $G = SU(2)$, it is known that the algebra of Weil polynomials is generated by the determinant of the algebra of matrices in $\mathfrak{su}(2)$, that is, $I^{SU(2)} = \mathbb{R}[\det]$. The determinant is a polynomial of degree two, hence the degree of every Weil polynomial is even. By Theorem 1, for $\dim M = 2k$, every aut $P$-invariant Lagrangian is of the type $\Lambda_f$ for $f \in I_k^{SU(2)}$. Then we can conclude that a $SU(2)$-bundle $\pi : P \to M$ has aut $P$-invariant Lagrangians different from zero if and only if $\dim M$ is a multiple of four. Hence the first dimension in which a non zero aut $P$-invariant Lagrangian density can be defined is $\dim M = 4$. This result agrees with the fact that every $SU(2)$-principal bundle over a manifold $M$ with $\dim M \leqslant 3$ is trivial (cf. [12, Chapter 2, Theorem 7.1]).

## 4. Variationally trivial Lagrangians

In the previous section it was proved that every aut $P$-invariant Lagrangian is variationally trivial. One can ask if the converse is true, namely, if every natural (that is gauge invariant) Lagrangian which is variationally trivial is aut $P$-invariant. It is known from the theory of characteristic classes that the only natural and functorial way for defining cohomology classes from the curvature form are the characteristic classes. As the aut $P$-invariance has already given the characteristic numbers $c_f$ for $f \in I_k^G$, $\dim M = 2k$, at first sight one can expect that the rest of the characteristic numbers must also appear in this context. This is the case as will be shown in Theorem 2: natural variationally trivial Lagrangians give not only all characteristic classes (not only those classes of degree $k$, $\dim M = 2k$) but also the other de Rham cohomology classes of $M$.

### *4.1. Notation*

Let $\mathcal{X}$ be an element of $\bigwedge^{2d} \mathfrak{X}(M) \otimes I_d^G \subset \bigwedge^{2d} \mathfrak{X}(M) \otimes S^d(\mathfrak{g}^*)$ (see Section 2.5) with $0 \leqslant d \leqslant [n/2]$, where $[\cdot]$ stands for the integer part. On the curvature bundle, define the function

$$\overline{\mathcal{X}}_d : \bigwedge^2 T^*M \otimes \operatorname{ad} P \to \mathbb{R},$$

by setting for any $w_2 \in (\bigwedge^2 T^*M \otimes \operatorname{ad} P)_x$, $x \in M$,

$$\overline{\mathcal{X}}_d(w_2) = \big\langle (\mathcal{X}_d)_x, w_2 \wedge \overset{(d)}{\cdots} \wedge w_2 \big\rangle, \tag{4.1}$$

where $\langle\, ,\rangle$ stands for the natural pairing between elements of $\bigwedge^{2d} \mathfrak{X}(M) \otimes I_d^G$ and its dual and $w_2 \wedge \overset{(d)}{\cdots} \wedge w_2 \in (\bigwedge^{2d} T^*M \otimes d \operatorname{ad} P)_x$. Remark that the action of a Weil polynomial $f \in I_d^G$ on $d \operatorname{ad} P$ is defined by means of formula (3.4).

In particular, if $\mathcal{X}_d = \chi \otimes f$ and $w_2 = \omega_2 \otimes \eta$, we have

$$\overline{\mathcal{X}}_k(w_2) = i_\chi \big(\omega_2 \wedge \overset{(d)}{\cdots} \wedge \omega_2\big) \cdot \bar{f}\big(\eta, \overset{(d)}{\ldots}, \eta\big).$$

Let

$$\mathcal{X}_d = \chi \otimes f = \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{2d}}} \lambda^{i_1,\ldots,i_{2d}}_{\alpha_1,\ldots,\alpha_d} B^{\alpha_1} \vee \cdots \vee B^{\alpha_d},$$

be the local expression of an element $\mathcal{X}_d \in \bigwedge^{2d} \mathfrak{X}(M) \otimes I_d^G$. In the natural coordinate system $(x^i, R_{ij}^\alpha)$, $1 \leqslant i < j \leqslant n$, $1 \leqslant \alpha \leqslant m$, on the curvature bundle, the expression of $\overline{\mathcal{X}_d}$ reads

$$\overline{\mathcal{X}_d} = \lambda_{\alpha_1,\ldots,\alpha_d}^{i_1,\ldots,i_{2d}} \sum_{\tau \in S_{2d}} \varepsilon(\tau) R_{i_{\tau(1)} i_{\tau(2)}}^{\alpha_1} \ldots R_{i_{\tau(2d-1)} i_{\tau(2d)}}^{\alpha_1}, \tag{4.2}$$

as shown by a direct computation.

### 4.2. *Characterization of the variational triviality*

**Theorem 2.** *Let $\pi : P \to M$ be a $G$-principal fiber bundle with $G$ connected and $M$ orientable. Let $v$ be a volume form on $M$. Then every gauge invariant variationally trivial Lagrangian density on $J^1C$ is of the form*

$$\mathcal{L} = \sum_{d=0}^{[n/2]} \overline{\mathcal{X}_d} \circ \Omega,$$

*where $\Omega$ is the curvature mapping and $\mathcal{X}_d \in \bigwedge^{2d} \mathfrak{X}(M) \otimes I_d^G$ are $I_d^G$ valued multi-vector fields which are 'divergence free', that is,*

$$L_{\mathcal{X}_d} v = 0, \tag{4.3}$$

*for all $d = 1, \ldots, [n/2]$.*

**Proof.** Let $\mathcal{L}v$ be a gauge invariant variationally trivial Lagrangian density. Then, by virtue of the Utiyama Theorem, there exists an adjoint invariant function

$$\overline{\mathcal{L}} : \bigwedge^2 T^*M \otimes \operatorname{ad} P \to \mathbb{R}$$

such that $\mathcal{L} = \overline{\mathcal{L}} \circ \Omega$. For the following local computations consider a chart $(U; x^1, \ldots, x^n)$ on $M$ such that $\pi^{-1}(U)$ is trivializable and $v = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. In the coordinate systems induced on $J^1C$ and on $\bigwedge^2 T^*M \otimes \operatorname{ad} P$ respectively, Proposition 1 yields, in this case, the following conditions for the Lagrangian $\mathcal{L}$:

$$\frac{\partial^2 \mathcal{L}}{\partial A_{i,h}^\alpha \partial A_{j,k}^\beta} + \frac{\partial^2 \mathcal{L}}{\partial A_{i,k}^\alpha \partial A_{j,h}^\beta} = 0, \quad i, j, k, h = 1, \ldots, n;\ \alpha, \beta = 1, \ldots, m, \tag{4.4}$$

$$\frac{\partial \mathcal{L}}{\partial A_j^\alpha} = \frac{\partial^2 \mathcal{L}}{\partial x^k \partial A_{j,k}^\alpha} + \frac{\partial^2 \mathcal{L}}{\partial A_i^\beta \partial A_{j,k}^\alpha} A_{i,k}^\beta, \quad j = 1, \ldots, n;\ \alpha = 1, \ldots, m. \tag{4.5}$$

On the other hand, taking the local expression of the curvature mapping (see formula (3.3)) and using the chain rule, yields

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{\partial \overline{\mathcal{L}}}{\partial x^k} \circ \Omega,$$

$$\frac{\partial \mathcal{L}}{\partial A_j^\alpha} = c_{\tau\alpha}^\beta A_i^\tau \frac{\partial \overline{\mathcal{L}}}{\partial R_{ji}^\beta} \circ \Omega,$$

$$\frac{\partial \mathcal{L}}{\partial A^\alpha_{j,k}} = \frac{\partial \overline{\mathcal{L}}}{\partial R^\alpha_{jk}} \circ \Omega.$$

With these formulas, Eqs. (4.4) and (4.5) can be rewritten as

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\alpha_{ih} \partial R^\beta_{jk}} \circ \Omega + \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\alpha_{ik} \partial R^\beta_{jh}} \circ \Omega = 0, \quad i, j, k, h = 1, \dots, n;\ \alpha, \beta = 1, \dots, m, \tag{4.6}$$

$$c^\beta_{\tau\alpha} A^\tau_i \frac{\partial \overline{\mathcal{L}}}{\partial R^\beta_{ji}} \circ \Omega = \frac{\partial^2 \overline{\mathcal{L}}}{\partial x^k \partial R^\alpha_{jk}} \circ \Omega + c^\gamma_{\tau\beta} A^\beta_{i,k} A^\tau_h \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ih} \partial R^\alpha_{jk}} \circ \Omega. \tag{4.7}$$

Remark that the composition with $\Omega$ can be eliminated in these equations, as the curvature mapping is surjective. The function $\overline{\mathcal{L}}$ is invariant under the adjoint representation in the curvature bundle. Locally, this invariance can be expressed as (cf. [9])

$$c^\gamma_{\tau\beta} R^\beta_{ik} \frac{\partial \overline{\mathcal{L}}}{\partial R^\gamma_{ik}} = 0, \quad \tau = 1, \dots, m. \tag{4.8}$$

Taking derivatives with respect to $R^\alpha_{jh}$, gives

$$2c^\gamma_{\tau\alpha} \frac{\partial \overline{\mathcal{L}}}{\partial R^\gamma_{jh}} + c^\gamma_{\tau\beta} R^\beta_{ik} \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ik} \partial R^\alpha_{jh}} = 0.$$

Multiplying by $A^\tau_h$ and summing yields

$$2c^\gamma_{\tau\alpha} A^\tau_h \frac{\partial \overline{\mathcal{L}}}{\partial R^\gamma_{jh}} + 2c^\gamma_{\tau\beta} A^\tau_h A^\beta_{i,k} \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ik} \partial R^\alpha_{jh}} - c^\gamma_{\tau\beta} c^\beta_{\mu\nu} A^\mu_i A^\nu_k A^\tau_h \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ik} \partial R^\alpha_{jh}} = 0.$$

The last summand vanishes. In fact, letting

$$I_\alpha = c^\gamma_{\tau\beta} c^\beta_{\mu\nu} A^\mu_i A^\nu_k A^\tau_h \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ik} \partial R^\alpha_{jh}}, \quad \alpha = 1, \dots, m,$$

one needs to prove that $I_\alpha = 0$, $\alpha = 1, \dots, m$. The Jacobi identity in the Lie algebra $\mathfrak{g}$

$$c^\gamma_{\tau\beta} c^\beta_{\mu\nu} + c^\gamma_{\nu\beta} c^\beta_{\tau\mu} + c^\gamma_{\mu\beta} c^\beta_{\nu\tau} = 0, \quad \gamma = 1, \dots, m$$

gives therefore

$$\begin{aligned}
I_\alpha &= -\left(c^\gamma_{\nu\beta} c^\beta_{\tau\mu} + c^\gamma_{\mu\beta} c^\beta_{\nu\tau}\right) A^\mu_i A^\nu_k A^\tau_h \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ik} \partial R^\alpha_{jh}} \\
&= -c^\gamma_{\nu\beta} c^\beta_{\tau\mu} A^\mu_i A^\nu_k A^\tau_h \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ik} \partial R^\alpha_{jh}} - c^\gamma_{\mu\beta} c^\beta_{\nu\tau} A^\mu_i A^\nu_k A^\tau_h \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ik} \partial R^\alpha_{jh}} \\
&= -c^\gamma_{\tau\beta} c^\beta_{\nu\mu} A^\mu_i A^\tau_h A^\nu_k \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ih} \partial R^\alpha_{jk}} - c^\gamma_{\tau\beta} c^\beta_{\nu\mu} A^\tau_h A^\nu_k A^\mu_i \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{hk} \partial R^\alpha_{ji}}.
\end{aligned}$$

As $R^\gamma_{hk} = -R^\gamma_{kh}$, Eq. (4.6) implies

$$I_\alpha = c^\gamma_{\tau\beta} c^\beta_{\nu\mu} A^\mu_i A^\tau_h A^\nu_k \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ik} \partial R^\alpha_{jh}} - c^\gamma_{\tau\beta} c^\beta_{\nu\mu} A^\tau_h A^\nu_k A^\mu_i \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^\gamma_{ki} \partial R^\alpha_{jh}} = 2I_\alpha,$$

thus concluding that $I_\alpha = 0$.

Therefore, using again formula (4.6), one concludes

$$c^{\gamma}_{\tau\alpha} A^{\tau}_{h} \frac{\partial \overline{\mathcal{L}}}{\partial R^{\gamma}_{jh}} = -c^{\gamma}_{\tau\beta} A^{\tau}_{h} A^{\beta}_{i,k} \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^{\gamma}_{ik} \partial R^{\alpha}_{jh}} = c^{\gamma}_{\tau\beta} A^{\tau}_{h} A^{\beta}_{i,k} \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^{\gamma}_{ih} \partial R^{\alpha}_{jk}}$$

and Eq. (4.7) can be simplified, thus obtaining

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial x^k \partial R^{\alpha}_{jk}} = 0, \quad j = 1, \ldots, n;\ \alpha = 1, \ldots, m. \tag{4.9}$$

This condition must be satisfied together with

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial R^{\alpha}_{ih} \partial R^{\beta}_{jk}} + \frac{\partial^2 \overline{\mathcal{L}}}{\partial R^{\alpha}_{ik} \partial R^{\beta}_{jh}} = 0, \quad i, j, k, h = 1, \ldots, n;\ \alpha, \beta = 1, \ldots, m. \tag{4.10}$$

Formula (4.10) gives then

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial R^{\alpha}_{ih} \partial R^{\alpha}_{ih}} = 0, \quad i, h = 1, \ldots, n;\ \alpha = 1, \ldots, m,$$

and therefore $\overline{\mathcal{L}} = A + R^{\alpha}_{ih} B$ for certain functions $A, B$, which are independent of the variable $R^{\alpha}_{ih}$. This condition, for $i, h$ and $\alpha$ arbitrary, is only possible if $\overline{\mathcal{L}}$ is a polynomial of the form

$$\overline{\mathcal{L}} = \sum_{d} \lambda^{i_1,\ldots,i_{2d}}_{\alpha_1,\ldots,\alpha_d} R^{\alpha_1}_{i_1 i_2} \ldots R^{\alpha_d}_{i_{2d-1} i_{2d}}, \tag{4.11}$$

where $\lambda^{i_1,\ldots,i_{2d}}_{\alpha_1,\ldots,\alpha_d} \in C^{\infty}(M)$. Using again (4.10) one concludes

$$\frac{\partial^2 \overline{\mathcal{L}}}{\partial R^{\alpha}_{ih} \partial R^{\alpha}_{jh}} = 0, \quad i, j, h = 1, \ldots, n;\ \alpha = 1, \ldots, m.$$

In other words, it is not possible to find a repeated index in every set $i_1, \ldots, i_{2d}$ of the sum in (4.11). This fact can be rephrased by saying that

$$\overline{\mathcal{L}} = \sum_{d=0}^{[n/2]} \sum_{\tau \in S_{2d}} \lambda^{\tau}_{\alpha_1,\ldots,\alpha_d} R^{\alpha_1}_{\tau(1)\tau(2)} \ldots R^{\alpha_d}_{\tau(2d-1)\tau(2d)}, \tag{4.12}$$

$\lambda^{\tau}_{\alpha_1,\ldots,\alpha_d} \in C^{\infty}(M)$, $\alpha_1, \ldots, \alpha_d \in \{1, \ldots, m\}$, $\tau \in S_{2d}$. Now, as was already done in the proof of Theorem 1, condition (4.10) means that

$$\lambda^{\tau}_{\alpha_1,\ldots,\alpha_d} = \varepsilon(\tau) \lambda^{I}_{\alpha_1,\ldots,\alpha_d}, \tag{4.13}$$

for every $\tau \in S_{2d}$. Thus the function $\overline{\mathcal{L}}$ can be expressed as

$$\overline{\mathcal{L}} = \sum_{d=0}^{[n/2]} \sum_{i_1 < \cdots < i_{2d}} \lambda^{i_1,\ldots,i_{2d}}_{\alpha_1,\ldots,\alpha_d} \sum_{\tau \in S_{2d}} \varepsilon(\tau) R^{\alpha_1}_{i_{\tau(1)} i_{\tau(2)}} \ldots R^{\alpha_d}_{i_{\tau(2d-1)} i_{\tau(2d)}}.$$

With the notation introduced in Section 4.1, it is easy to see that

$$\overline{\mathcal{L}} = \sum_{d=0}^{[n/2]} \overline{\mathcal{X}}_d,$$

$\mathcal{X}_d \in \bigwedge^{2d} \mathfrak{X}(M) \otimes S^d(\mathfrak{g}^*)$ being valued multi-vector fields whose expressions are

$$\mathcal{X}_d = \sum_{i_1<\cdots<i_{2d}} (-1)^{|I|+2d} \lambda^{i_1,\dots,i_{2d}}_{\alpha_1,\dots,\alpha_d} \frac{\partial}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_{2d}}} \otimes B^{\alpha_1} \vee \cdots \vee B^{\alpha_d},$$

where $|I| = i_1 + \cdots + i_{2d}$. The condition of adjoint invariance of $\overline{\mathcal{L}}$ easily yields the condition of adjoint invariance for each valued multi-vector field $\mathcal{X}_d$; that is, $\mathcal{X}_d \in \bigwedge^{2d} \mathfrak{X}(M) \otimes I^G_d$ for every $d = 1, \dots, [n/2]$. It remains to see whether condition (4.3) is fulfilled. We have

$$\begin{aligned} L_{\mathcal{X}_d} v &= \mathrm{d}\big(\lambda^{i_1,\dots,i_{2d}}_{\alpha_1,\dots,\alpha_d} i_{\frac{\partial}{\partial x^{i_1}} \wedge\cdots\wedge \frac{\partial}{\partial x^{i_{2d}}}} v\big) \otimes B^{\alpha_1} \vee \cdots \vee B^{\alpha_d} \\ &= \frac{\partial(\lambda^{i_1,\dots,i_{2d}}_{\alpha_1,\dots,\alpha_d})}{\partial x^{i_s}} \mathrm{d}x^{i_s} \wedge \big(i_{\frac{\partial}{\partial x^{i_1}} \wedge\cdots\wedge \frac{\partial}{\partial x^{i_{2d}}}} v\big) \otimes B^{\alpha_1} \vee \cdots \vee B^{\alpha_d}. \end{aligned}$$

Fix an arbitrary set of indices $1 \leqslant j_2 < \cdots < j_{2d} \leqslant n$. The coefficient of the summand $i_{\frac{\partial}{\partial x^{j_2}} \wedge\cdots\wedge \frac{\partial}{\partial x^{j_{2d}}}} v$ in the previous formula is

$$\sum_l \frac{\partial(\lambda^{l,j_2,\dots,j_{2d}}_{\alpha_1,\dots,\alpha_d})}{\partial x^l} \otimes B^{\alpha_1} \vee \cdots \vee B^{\alpha_d}, \tag{4.14}$$

where

$$\lambda^{l,j_2,\dots,j_{2d}}_{\alpha_1,\dots,\alpha_d} = \varepsilon(\tau) \lambda^{\tau(l),\tau(j_2),\dots,\tau(j_{2d})}_{\alpha_1,\dots,\alpha_d},$$

$\tau$ being the only permutation of the set $\{l, j_2, \dots, j_{2d}\}$ such that $\tau(l) < \tau(j_2) < \cdots < \tau(j_{2d})$, whenever the indices $l, j_2, \dots, j_{2d}$ are not increasing. Hence, $L_{\mathcal{X}_d} v = 0$ if and only if expression (4.14) vanishes for every choice of indices $1 \leqslant j_2 < \cdots < j_{2d} \leqslant n$. On the other hand, taking into account expression (4.12) of $\overline{\mathcal{L}}$ and the property (4.13), condition (4.9) reads

$$\begin{aligned} \frac{\partial^2 \overline{\mathcal{L}}}{\partial x^l \partial R^\alpha_{jl}} &= \sum_{d=0}^{[n/2]} \frac{\partial(\lambda^{i_1,\dots,i_{2d}}_{\alpha_1,\dots,\alpha_d})}{\partial x^l} \frac{\partial R^{\alpha_1}_{i_1 i_2} \dots R^{\alpha_d}_{i_{2d-1} i_{2d}}}{\partial R^\alpha_{jl}} \\ &= \sum_{d=0}^{[n/2]} d \frac{\partial(\lambda^{j,l,\dots,i_{2d}}_{\alpha,\alpha_2,\dots,\alpha_d})}{\partial x^l} R^{\alpha_2}_{i_3 i_4} \dots R^{\alpha_d}_{i_{2d-1} i_{2d}}. \end{aligned}$$

The left hand side vanishes if and only if

$$\sum_l \frac{\partial(\lambda^{j,l,\dots,i_{2d}}_{\alpha,\alpha_2,\dots,\alpha_d})}{\partial x^l} = 0,$$

for every choice of the indices $\alpha, \alpha_2, \dots, \alpha_d = 1, \dots, m$ and $j, i_3, \dots, i_{2d} = 1, \dots, n$. This condition is equivalent to (4.14), thus finishing the proof. □

**Remark 2.** The variational triviality of the Lagrangians $\Lambda_f$ defined by Weil polynomials $f \in I^G_k$, $\dim M = 2k$, can be obtain from Theorem 2, without using the theory of the Weil polynomials, as we have done in Corollary 1. Indeed, given $f$, consider the element $\mathcal{X}_f = \chi \otimes f \in \bigwedge^n T^*M \otimes I^G_k$, where $\chi$ is the dual of the volume form $v$, that is, the unique section of $\bigwedge^n TM$ such that $i_\chi v = 1$. It is easy to see that $\overline{\mathcal{X}}_f \circ \Omega = \Lambda_f$. Hence, as $L_{\mathcal{X}} v = \mathrm{d} i_\chi v \otimes f = 0$, we obtain the variational triviality of $\Lambda_f$ by virtue of Theorem 2.

In Theorem 1, aut $P$-invariance gave topological information on the bundle $\pi : P \to M$ by means of the characteristic numbers $c_f$, $f \in I_k^G$, $\dim M = 2k$. Theorem 2 gives additional information. Indeed, let $\mathcal{X} = \chi \otimes f \in \bigwedge^{2d} T^*M \otimes I_d^G$ be an $I^G$-valued multi-vector field which is 'divergence free'. We have

$$L_{\mathcal{X}} v = \mathrm{d}(i_\chi v) \otimes f = 0,$$

that is, the form $i_\chi v$ is closed and defines a cohomology class of degree $n - 2d$, for $d = 0, \ldots, [n/2]$. In fact, there is a bijective correspondence of closed forms and 'divergence free' multi-vector fields. Roughly speaking, variationally trivial natural Lagrangians are characterized by de Rham classes of the base manifold $M$ of the same parity like $\dim M$ and the characteristic classes of the bundle $\pi : P \to M$ of arbitrary degree.

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